# NEARLY GORENSTEIN AND ALMOST GORENSTEIN 2-DIMENSIONAL NORMAL LOCAL RINGS

TOMOHIRO OKUMA (YAMAGATA UNIV.), KEI-ICHI WATANABE (MEIJI UNIVERSITY AND NIHON UNIVERSOTY), KEN-ICHI YOSHIDA (COLLEGE OF HUMANITY AND SCIENCES, NIHON UNIVERSITY)

This talked is based on our joint work in progress (partially [3]).

The concepts of almost Gorenstein rings (resp. nearly Gorenstein rings) were proposed in [1] (resp. [2]) to find Cohen-Macaulay rings which are "near" to Gorenstein rings.

In this talk, we want to find the condition for a 2-dimensional local ring or a 2-dimensional normal graded ring $R = \bigoplus_{n\geq 0} R_n$ to be nearly Gorenstein using either resolution of singularities $f : X \to \mathrm{Spec}(R)$ or DDP (Demazure-Dolgachev-Pinkham) construction of $R$.

## 1. Definitions

Let $(A, \mathfrak{m})$ be a Noethrian local ring. **We assume that $A$ is Cohen-Macaulay and let $K_A$ be the canonical module of $A$.**

**Definition 1.1.** (1) We write $\mathrm{Tr}_A(K_A) = (f(x) \mid x \in K_A \text{ and } f \in \mathrm{Hom}_A(K_A, A))$ and call it the **canonical trace ideal** of $A$. Note that $\mathrm{Tr}_A(K_A) = A$ if and only of $K_A$ is a free $A$-module.

(2) We say that $A$ is **nearly Gorenstein** if $\mathfrak{m} \subset \mathrm{Tr}_A(K_A)$. By definition, Gorenstein ringes are nearly Gorenstein.

(3) Put $\dim A = d$. We say that $A$ is almost Gorenstein if for a *general* element $\omega \in K_A$, $K_A/A\omega$ is an Ulrich $A$-module of dimension $d - 1$. It is easy to show that an almost Gorenstein ring of *dimension* 1 is nearly Gorenstein.

(4) If $U$ is a finitely generated $A$-module of dimension $d$, we say that $U$ is an **Ulrich** $A$-module if multiplicity of $M$ is equal to $\mu_A(M) := \dim_{A/\mathfrak{m}} M/\mathfrak{m}M$.

## 2. Computation of Canonical Trace ideal of a 2-dimensional normal ring via resolution of singularities

Let $(A, \mathfrak{m}, k)$ be an excellent two-dimensional normal local domain containing an algebraically closed field isomorphic to the residue field $k$ and let $K_A$ denote a canonical module of $A$. Let $\pi \colon X \to \mathrm{Spec}(A)$ be a resolution of singularities with exceptional set $\mathbb{E} := \pi^{-1}(\mathfrak{m})$ and $\mathbb{E} = \bigcup_{i=1}^n E_i$ the decomposition into the irreducible components. An element $Z \in \oplus_{i=1}^r \mathbb{Z}E_i$ is called a *cycle* and if $Z = \sum_{i=1}^r n_i E_i$, we say $Z > 0$ if all $n_i \geq 0$ and $Z \neq 0$.

**Definition 2.1.** (1) Intersection theory on $\mathbb{Z}\mathbb{E}$ is defined and ite is known that the intersection matrix $(E_iE_j)_{i,j=1}^r$ is **negative definite**.

(1) a cycle $Z > 0$ is called **anti-nef** if $ZE_i \leq 0$ for every $E_i$.

(2) There exists *minimal anti-nef* cycle and we call it the **fundamental cycle** and denote it by $Z_f$.

(3) We call a sequence $W = Y_0 < Y_1 < \ldots < Y_N$ a **computation sequence** from $W$ if for every $i$, $Y_{i+1} = Y_i + E_{j_i}$ with $Y_i E_{j_i} > 0$ and $Y_N$ is anti-nef. There are many choices of computation sequences from $W$, but every computation sequence terminates at the minimal anti-nef cycle $> W$.

**Definition 2.2.** Let $L$ be a divisor on $X$. Since $X$ is regular, $\mathcal{O}_X(L)$ is an invertible $\mathcal{O}_X$-module. A cycle $W > 0$ is a fixed component of $L$ if the natural inclusion $\mathcal{O}_X(L - W) \subset \mathcal{O}_X(L)$ induces bijection $H^0(X, \mathcal{O}_X(L - W)) = H^0(X, \mathcal{O}_X(L))$. If for every $E_i$, $H^0(X, \mathcal{O}_X(L - E_i) \subsetneq H^0(X, \mathcal{O}_X(L))$, we say that $L$ is fixed-cimponent free. The maximal fixed component of $L$ is called the **fixed component** of $L$.

We denote by $K_X$ denote the canonical divisor on $X$. If $A$ is a rational, then $K_X$ is fixed component free and $H^0(X, \mathcal{O}_X(K_X)) = K_A$. (If $A$ is not rational, we must add $C_X$ to have $K_A = H^0(X, \mathcal{O}_X(K_X + C_X))$.

**Proposition 2.3.** *Assume that $L$ has no fixed components and let $F \geq 0$ be the fixed component of $-L$. Then we have $\mathrm{Tr}_{H^0(\mathcal{O}_X(L))}(A) \subset H^0(\mathcal{O}_X(-F))$.*

The main result of this talk is the following Theorem.

**Theorem 2.4.** *If $A$ is a rational singularity, then $A$ is nearly Gorenstein if and only if $K_X + Z_f$ is anti-nef. Otherwise, if a computation sequence starting from $K_X + Z_f$ terminates at $K_X + W$, then $\mathrm{Tr}_A(K_A) = H^0(X, \mathcal{O}_X(-W))$.*

Using this Theorem, we can classify the resolution graph of nearly Gorenstein rational singularities in the case the graph is "star-shaped".

## 3. Nearly Gorenstein Normal Graded Rings.

Let $R = \bigoplus_{n \geq 0} R_n$ be a **normal graded** ring of dimension $d \geq 2$ finitely generated over a field $R_0 = k$. Then the canonical module $K_R$ of $R$ has a natural structure of a graded $R$-module and also is a reflexive $R$-module of rank 1. Then we can consider $K_R$ as a graded submodule of $Q(R) = S^{-1}R$, where $S$ is the set of all nonzero homogeneous elements of $R$. We fix an embedding of $K_R \subset Q(R)$ as a graded fractional ideal of $R$. Then $\mathrm{K}_{\mathrm{R}}^{-1}$ is generated by all homogeneous elements $\phi \in Q(R)$ such that $\phi K_R \subset R$ and $R$ is nearly Gorenstein if and only if

$$\mathfrak{m}_R = R_+ \subset K_R \,\mathrm{K}_{\mathrm{R}}^{-1}.$$

Since $K_R$ and $\mathrm{K}_{\mathrm{R}}^{-1}$ can be precisely described by geometric data of $X = \mathrm{Proj}(R)$ and $K_X, -K_X$, we can determine the nearly Gorenstein properties of normal graded rings in many examples.

In particular, we can show that 2-dimensional normal local (or graded) graded rings, which are almost Gorenstein **and** nearly Gorenstein are very few if $R$ is not rational or elliptic.

## References


[1] Shiro Goto, Ryo Takahashi, and Naoki Taniguchi, *Almost Gorenstein rings—towards a theory of higher dimension*, J. Pure Appl. Algebra **219** (2015), no. 7, 2666–2712.

[2] Jürgen Herzog, Takayuki Hibi, and Dumitru I. Stamate, *The trace of the canonical module*, Israel J. Math. **233** (2019), no. 1, 133–165.

[3] Tomohiro Okuma, Kei-ichi Watanabe, and Ken-ichi Yoshida, *A geometric description of almost Gorensteinness for two-dimensional normal singularities*, arXiv:2410.23911.